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A FINITE ELEMENT SOLUTION OF THE COUPLED DYNAMICAL THERMOELASTI--ETC(U)
JAN 80 Y CHEN, H GHONEIM, J DAVIS DAAG29-79-G-0052
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A FINITE ELEMENT SOLUTION OF THE COUPLED DYNAMICAL THERMOELASTICITY EQUATIONS IN A SLAB

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ABSTRACT

A finite-element methodology has been presented to solve the one-dimensional couple dynamical thermoelasticity. The spatial variable is discretized by the Galerkin scheme and the central difference scheme. The time variable is discretized by the Galerkin and the backward difference scheme. It was found that the two spatial discretization approaches give close comparison in accuracy when used in conjunction with the backward difference in time. To simulate the wave-like behavior in micro-time scale becomes expensive but there is no technical difficulty. To simulate the macro-time phenomena, the micro-time behavior would disappear.

Two test problems are presented to show the micro- and the macro-time behaviors of the thermoelastic slab. The responses to a realistic set of pressure and temperature pulse inputs simulating the interior ballistic type of action are also presented.

INTRODUCTION

The coupled thermoelasticity problem has been treated by many investigators in the past. A partial listing of the relevant papers are given in the reference [1-5], most of these papers deal with the problem of a semi-infinite space and a few contain a comprehensive numerical treatment. In many applied problems the coupling term in the energy balance equation is ignorable due to the smallness of its magnitude. In other problems such as that of the thermal stresses in a gun barrel, due to the high rate of loading the coupling must be kept until the results of the analysis shows otherwise. Also, the inertial effects might be important and should be kept in the analysis for such problems. The particular mathematical model to be dealt in this paper is a boundary value problem for a pair of partial differential equations governing the stress and the temperature fields. The boundary conditions correspond to the time dependent stress and temperature acting on one surface and the free traction and ambient temperature on the other surface.

A finite element approach is used in discretizing the spatial as well as the time variable. In each discretization a choice of two different approximation is provided to furnish a comparison. Although the problem analyzed is one-dimensional in space the methodology developed can be transferred to the problem of an annulus without too much change in its principles and techniques. In the next section the mathematical model will be given in a general form first and then specialized to a slab.

MATHEMATICAL MODEL

Let the temperature in the medium be T and its displacement be denoted by $\underline{u}$, then the equation of motion of the medium is given in a vectorial form as follows.

$$\mu\nabla^2\underline{u}+(\lambda+\mu)\text{grad div }\underline{u}+\underline{F}-\gamma\text{grad}T=\rho\ddot{\underline{u}} \qquad (1)$$

where ∇^2 is the Laplacian, λ and μ are the Lame's constants, $\underline{F}$ is the body force vector, ρ is the density and γ is a constant given by

$$\gamma=(3\lambda+2\mu)\alpha \qquad (2)$$

where α is the coefficient of thermal expansion. The last term in the left-hand side of Eq. (1) is the force per unit volume due to the temperature gradient in the medium. In an uncoupled problem the temperature field is given.

The second equation of the pair of partial differential equations of the moadel is the energy balance as given by [1]

$$\nabla^2T-\frac{1}{\kappa}\dot{T}-\eta\text{ div }\dot{\underline{u}}=-\frac{Q}{\kappa} \qquad (3)$$

where κ is the diffusivity, Q is the heat source and $\eta=\gamma T_0/\rho c\kappa$, T_0 is the ambient temperature.

Assume that the body force $\underline{F}$ and the heat source Q can be dropped and that the displacement vector $\underline{u}$ is of the form

$$\underline{u}=\text{grad }\phi+\text{curl }\underline{\psi} \qquad (4)$$

Substituting (4) into (1) and (3) yields a pair of couple PDE in ϕ and T and an uncoupled PDE in ψ describing the shear wave propagation. These equations are:

$$\nabla^2T-\frac{1}{\kappa}\dot{T}-\eta\nabla^2\dot{\phi}=0 \qquad (5)$$

$$\nabla^2\phi-\frac{1}{C_1^2}\ddot{\phi}=mT \qquad (6)$$

$$\nabla^2\underline{\psi}-\frac{1}{C_2^2}\ddot{\underline{\psi}}=0 \qquad (7)$$

where $m=\gamma(\lambda+2\mu)$, $C_1^2=(\lambda+2\mu)/\rho$ and $C_2^2=\mu/\rho$, C_1 and and C_2 are the dilatational and the shear wave speed respectively. Since the temperature fields has no interaction with the shear wave, the subsequent analysis will direct its attention to (5) and (6).

To specialize the pair of equations to a slab we can write (6) in the form

$$\phi_{xx}-\frac{1}{C_1^2}\phi_{tt}=mT \qquad (8)$$

and (5) as follows

$$T_{xx}-\frac{1}{\kappa}T_t-\eta\phi_{xx}=0 \qquad (9)$$

Since it is more convenient to use the stress instead the potential ϕ as the variable in one-dimensional problem, we introduce the constitutive equation for a thermoelastic medium,

$$\sigma = (\lambda+2\mu)u_{xx} - (3\lambda+2\mu)\alpha T \tag{10}$$

where σ is the uniaxial stress.

Using (10) to eliminate u from (8) yields

$$C_1^2\sigma_{xx} - \sigma_{tt} - \gamma T_{tt} = 0 \tag{11}$$

and

$$T_{xx} - \left(\frac{1}{\kappa} + \eta m\right)T_t - \left(\frac{\eta}{\lambda+2\mu}\right)\sigma_t = 0 \tag{12}$$

The boundary conditions are given as follows.

$x = o$, $\sigma(0,t) - f^*(t) = 0$

$$T_x(0,t) - \beta_1^*(T(0,t) - T_o) - (g^*(t) - T_o) = 0 \tag{13}$$

$x = l$, $\sigma(l,t) = 0$

$$T_x(l,t) + \beta_2^*(T(l,t) - T_o) = 0 \tag{14}$$

where l is the thickness of the slab, $\beta_1^* = h_1/K$, $\beta_2^* = h_2/K$, h_1 and h_2 are the heat transfer coefficient and K the conductivity of the material.

NON-DIMENSIONAL EQUATIONS

To facilitate computation the following non-dimensional quantities are introduced

$$\bar{T} = \frac{T-T_o}{T_o}, \quad \bar{x} = \frac{x}{l}, \quad \bar{t} = \frac{\kappa t}{l^2}, \quad \bar{\sigma} = \frac{\sigma}{\lambda+2\mu} \tag{15}$$

After converting the equations into the non-dimensional form the bars are dropped for ease of writing.
Thus, after some algebra, we have

$$\sigma_{xx} - \sigma_{tt} - \lambda_2 T_{tt} = 0 \tag{16}$$

and

$$\frac{1}{D} T_{xx} - (1+\lambda_1\lambda_2)T_t - \lambda_1\sigma_t = 0 \tag{17}$$

where $D = C_1 l/\kappa$, $\lambda_1 = \left(\frac{3\lambda+2\mu}{\rho c}\right)\alpha$, $\lambda_2 = \left(\frac{3\lambda+2\mu}{\lambda+2\mu}\right)\alpha\, T_o$.

The boundary conditions reduce to

$x = 0$, $\sigma(0,t) - f(t) = 0$

$$T_x - \beta_1 T - \beta_1 g = 0 \tag{18}$$

$x = 1$ $\sigma(1,t) = 0$

$$T_x - \beta_2 T = 0 \tag{19}$$

where $\beta_1 = \beta_1^* l$, $\beta_2 = \beta_2^* l$, $f(t) = f^*/(\lambda+2\mu)$, $g = \frac{g^* - T_o}{T_o}$.

THE SPACE DOMAIN

To discretize the spatial variable in (16) and (17) we used two approaches, i.e. the Galerkin procedure and the central difference scheme.

The Galerkin Procedure

In this procedure we shall assume that both approximate solutions of (16) and (17) $\tilde{\sigma}$ and $\tilde{T}$ can be expanded in terms the same spatial approximate function $\psi_i(x)$ in the following way.
Let

$$\tilde{\sigma} = \sum_{i=1}^{n} \psi_i(x)\sigma_i(t) + \psi_o(x)\sigma_o(t)$$
$$\tilde{T} = \sum_{i=0}^{N} \psi_i(x)T_i(t) \tag{20}$$

where $N = n+1$, n is the number of internal nodes and $\psi_i(x)$ is defined as

$$\psi_i(x) \begin{cases} = 1 - \left|\dfrac{x-x_i}{h}\right|, & |x-x_i| \leq h \\ = 0, & |x-x_i| \geq h \end{cases} \quad i=0,1,2,\ldots N \tag{21}$$

where h is dimensionless length of the interval. A graphical illustration of the function in (21) is given in Figure 1.

For convenience we introduce the symbols L_1 and L_2 such that the PDE in (16) and (17) can be abbreviated as

$$L_1(\sigma,T) = 0,$$
$$L_2(\sigma,T) = 0. \tag{22}$$

Then the residue R_1 and R_2 can be defined to be

$$R_1 \equiv L_1(\tilde{\sigma},\tilde{T}) \quad \text{and}$$
$$R_2 \equiv L_2(\tilde{\sigma},\tilde{T}). \tag{23}$$

For the boundary conditions we abbreviate (18) and (19) as

$$B_1(\sigma) = 0, \quad B_1(T) = 0 \quad \text{and} \tag{24}$$

$$B_2(\sigma) = 0, \quad B_2(T) = 0 \tag{25}$$

In the Galerking procedure we require that the weighted residue to vanish with the weighting function taken as $\psi_j(x)$, i.e.

$$\int_0^1 R_i\psi_j(x)\,dx = 0 \quad i = 1,2 \quad j = 0,1,2,\cdots,N \tag{26}$$

For $i = 1$, (26) leads to after integration by parts and some algebra,

$$\frac{1}{C}\int_0^1 \sum_{i=0}^{N} \nu_i'(x)\nu_j'(x)T_i(t)\,dx$$

$$+(1+\lambda_1\lambda_2)\int_0^1 \sum_{i=0}^{N} \nu_i(x)\nu_j(x)\dot{T}_i(t)\,dx$$

$$+\lambda_1\int_0^1 \sum_{i=0}^{N} \nu_i(x)\nu_j(x)\dot{\sigma}_i(t)\,dx = -\lambda_1\int_0^1 \nu_0(x)\nu_j(x)\dot{\sigma}_0(t)\,dx$$

$$-\frac{1}{C}\Big[\sum_{i=0}^{N} \nu_i'(x)\nu_j(x)T_i(t)\,dx\Big]_0^1 \qquad (27)$$

To treat the term in the bracket in (27) the thermal boundary conditions in (24) and (25) will be used. Let

$$R(0) \equiv B_1(\tilde{T})$$

$$= \sum_{i=0}^{N} \nu_i'(x)T_i(t) - \beta_1 \sum_{i=0}^{N} \nu_i(x)T_i(t) + \beta_1 g(t) \qquad (28)$$

and

$$R(1) \equiv B_2(\tilde{T})$$

$$= \sum_{i=0}^{N} \nu_i'(x)T_i(t) + \beta_2 \sum_{i=0}^{N} \nu_i(x)T_i(t) \qquad (29)$$

We demand that the weighted residue of $R(0)$ and $R(1)$ vanish. This statement, when (28) and (29) are used, can be casted into the following form by introducing a delta function in each term.

$$\int_0^1 \sum_{i=0}^{N} \nu_i'(x)\nu_j(x)T_i(t)\delta(x)\,dx$$

$$-\beta_1\int_0^1 \sum_{i=0}^{N} \nu_i(x)\nu_j(x)T_i(t)\delta(x)\,dx$$

$$+\beta_1\int_0^1 \nu_j(x)\delta(x)g(t)\,dx = 0 \qquad (30)$$

and

$$\int_0^1 \sum_{i=0}^{N} \nu_i'(x)\nu_j(x)T_i(t)\delta(x-1)\,dx$$

$$+\beta_2\int_0^1 \sum_{i=0}^{N} \nu_i(x)\delta(x-1)T_i(t)\,dx = 0 \qquad (31)$$

Finally,

$$\Big[\sum_{i=0}^{N} \nu_i(x)\nu_j(x)T_i(t)\Big]_0^1 =$$

$$-\beta_1\int_0^1 \sum_{i=0}^{N} \nu_i(x)\nu_j(x)T_i(t)\delta(x)\,dx$$

$$+\beta_1\int_0^1 \sum_{i=0}^{N} \nu_i(x)\delta(x)g(t)\,dx$$

$$-\beta_2\int_0^1 \sum_{i=0}^{N} \nu_i(x)\delta(x-1)T_i(t)\,dx \qquad (32)$$

Performing the necessary integration in (27) and (32) yields the x-discretized equation of (11).

$$\frac{1}{Ch^2}\begin{bmatrix} (1+\beta_1 h) & -1 & & & & \\ -1 & 2 & -1 & & & \\ & -1 & 2 & -1 & & \\ & & - & - & - & \\ & & & -1 & 2 & -1 \\ & & & & -1 & (1+\beta_2 h) \end{bmatrix}\begin{bmatrix} T_0 \\ T_1 \\ T_2 \\ \vdots \\ T_n \\ T_N \end{bmatrix}$$

$$+(1+\lambda_1\lambda_2)\begin{bmatrix} \frac{2}{6} & \frac{1}{6} & & & & \\ \frac{1}{6} & \frac{4}{6} & \frac{1}{6} & & & \\ & \frac{1}{6} & \frac{4}{6} & \frac{1}{6} & & \\ & & - & - & - & \\ & & & \frac{1}{6} & \frac{4}{6} & \frac{1}{6} \\ & & & & \frac{1}{6} & \frac{4}{6} \end{bmatrix}\begin{bmatrix} \dot{T}_0 \\ \dot{T}_1 \\ \dot{T}_2 \\ \vdots \\ \dot{T}_n \\ \dot{T}_N \end{bmatrix}$$

$$+\lambda_1\begin{bmatrix} \frac{1}{6} & & & \\ \frac{4}{6} & \frac{1}{6} & & \\ \frac{1}{6} & \frac{4}{6} & \frac{1}{6} & \\ & - & - & - \\ & & \frac{1}{6} & \frac{4}{6} \\ & & & \frac{1}{6} \end{bmatrix}\begin{bmatrix} \dot{\sigma}_1 \\ \dot{\sigma}_2 \\ \dot{\sigma}_3 \\ \vdots \\ \vdots \\ \dot{\sigma}_n \end{bmatrix} = \begin{bmatrix} -\frac{\lambda_1}{3}\dot{f} + \frac{\beta_1}{Ch}g \\ \frac{\lambda_1}{6}\dot{f} \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}$$

(33)

For i = 2, $\int_0^1 R_2\psi_j(x)dx = 0$

By a similar calculation as above we obtain the x-discretized equation of (12)

$$\frac{1}{h^2}\begin{bmatrix} 2 & 1 & & & & \\ -1 & 2 & -1 & & & \\ & -1 & 2 & -1 & & \\ & & \cdots & \cdots & & \\ & & & -1 & 2 & -1 \\ & & & & -1 & 2 \end{bmatrix}\begin{bmatrix}\sigma_1\\ \sigma_2\\ \sigma_3\\ \vdots\\ \sigma_{n-1}\\ \sigma_n\end{bmatrix} + \begin{bmatrix} \frac{4}{6} & \frac{1}{6} & & & & \\ \frac{1}{6} & \frac{4}{6} & \frac{1}{6} & & & \\ & \frac{1}{6} & \frac{4}{6} & \frac{1}{6} & & \\ & & \cdots & \cdots & & \\ & & & \frac{1}{6} & \frac{4}{6} & \frac{1}{6} \\ & & & & \frac{1}{6} & \frac{4}{6} \end{bmatrix}\begin{bmatrix}\ddot\sigma_1\\ \ddot\sigma_2\\ \ddot\sigma_3\\ \vdots\\ \ddot\sigma_n\\ \ddot\sigma_n\end{bmatrix}$$

$$+\lambda_2\begin{bmatrix} \frac{1}{6} & \frac{4}{6} & \frac{1}{6} & & & & \\ & \frac{1}{6} & \frac{4}{6} & \frac{1}{6} & & & \\ & & \frac{1}{6} & \frac{4}{6} & \frac{1}{6} & & \\ & & & \cdots & \cdots & & \\ & & & & \frac{1}{6} & \frac{4}{6} & \frac{1}{6} \\ & & & & & \frac{1}{6} & \frac{4}{6} & \frac{1}{6} \end{bmatrix}\begin{bmatrix}\ddot T_o\\ \ddot T_1\\ \ddot T_2\\ \vdots\\ \ddot T_n\\ \ddot T_N\end{bmatrix} = \begin{bmatrix}\frac{1}{h^2}f - \frac{1}{6}\ddot f\\ 0\\ 0\\ \vdots\\ 0\\ 0\end{bmatrix} \tag{34}$$

Equations (33) and (34) can be abbreviated as follows

$$\frac{1}{Ch^2}[A_1]\underline{T} + (1+\lambda_1\lambda_2)[B_1]\underline{\dot T} + \lambda_1[H]\underline{\dot\sigma} = \underline{f_1} \tag{35}$$

$$\frac{1}{h^2}[A_2]\underline{\sigma} + [B_2]\underline{\ddot\sigma} + \lambda_2[H]^T\underline{\ddot T} = \underline{f_2} \tag{35}$$

where $[A_1]$, $[A_2]$, etc. are the coefficient matrices in (33) and (34).

Central Difference Scheme

In this scheme the approximating functions $\psi_1(x)$ are quadratic instead of linear as in the case of the Galerkin scheme. The weighting function is taken to be $\delta(x-x_j)$.

Let

$$\tilde\sigma(x,t) = \sum_{i-1}^{i+1}\psi_j(x)\sigma_j(t) \tag{36}$$

$$\tilde T(x,t) = \sum_{i-1}^{i+1}\psi_j(x)T_j(t) \tag{36}$$

where

$$\psi_{j-1}(x) = \frac{(x-x_j)(x-x_{j+1})}{(x_{j-1}-x_j)(x_{j-1}-x_{j+1})}$$

$$\psi_j(x) = \frac{(x-x_{j-1})(x-x_{j+1})}{(x_j-x_{j-1})(x_j-x_{j+1})} \tag{37}$$

$$\psi_{j+1}(x) = \frac{(x-x_{j-1})x-x_j)}{(x_{j+1}-x_{j-1})(x_{j+1}-x_j)}$$

A graphical sketch of (37) is given in Figure 2

The residues are defined as:

$$R_1 \equiv L_1(\tilde\sigma, \tilde T) \quad \text{and} \quad R_2 \equiv L_2(\tilde\sigma, \tilde T) \tag{38}$$

and we require that

$$\int_0^1 R_j\delta(x-x_i)dx = 0, \qquad j = 1,2. \tag{39}$$

Performing the operation specified in (39) yields

$$\begin{aligned}&\frac{1}{Ch^2}(-T_{i-1}+2T_i-T_{i+1}) + (1+\lambda_1\lambda_2)\dot T_i + \lambda_1\dot\sigma_i = 0\\ &\frac{1}{h^2}(-\sigma_{i-1}+2\sigma_i-\sigma_{i+1}) + \ddot\sigma_i + \lambda_2\ddot T_i = 0\end{aligned} \tag{40}$$

The boundary conditions will be discretized as follows. Applying (40) to the boundaries at x=0, we have

$$\frac{1}{Ch^2}(-T_{-1}+2T_o-T_1) + (1+\lambda_1\lambda_2)\dot T_o + \lambda_1\dot\sigma_o = 0 \tag{41}$$

Calculating the residue R(0),

$R(0) \equiv B_1(T)$

and setting

$$\int_0^1 R(0)\,\delta(x-0)dx = 0$$

result

$$\frac{1}{2h}(-T_{-1}+T_1) - \beta_1 T_o + \beta_1 g = 0$$

from which we can calculate T_{-1},

$$T_{-1} = T_1 - 2h\beta_1 T_o + 2h\beta_1 g \tag{42}$$

Substituting (42) into (41) yields

$$\frac{1}{Ch^2}[2T_o(1+h\beta_1)-2T_1] + (1+\lambda_1\lambda_2)\dot{T}_o = \frac{2\beta_1}{Ch}\, g - \lambda_1 \dot{f} \tag{43}$$

Similarly, setting $\int_0^1 R(1)\delta(x-1)dx=0$, we have

$$\frac{1}{Ch^2}[-2T_{N-1}+2(1+\beta_2 h)T_N] + (1+\lambda_1\lambda_2)T_N = 0 \tag{44}$$

From (40, (42) and (44) we finally obtain the x-discretized equations.

$$\frac{1}{Ch^2}\begin{bmatrix} (1+\beta_1 h) & -1 & & & & \\ -1 & 2 & -1 & & & \\ & -1 & 2 & -1 & & \\ & & - - - & - - - & & \\ & & & -1 & 2 & -1 \\ & & & & -1 & (1+\beta_2 h) \end{bmatrix} \begin{bmatrix} T_o \\ T_1 \\ T_2 \\ \vdots \\ T_n \\ T_N \end{bmatrix} + (1\lambda_1\lambda_2) \begin{bmatrix} \frac{1}{2} & & & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & & - - - & & \\ & & & & 1 & \\ & & & & & \frac{1}{2} \end{bmatrix} \begin{bmatrix} \dot{T}_o \\ \dot{T}_1 \\ \dot{T}_2 \\ \vdots \\ \dot{T}_n \\ \dot{T}_N \end{bmatrix}$$

$$+\lambda_2 \begin{bmatrix} 0 & & & \\ 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \\ & & & 0 \end{bmatrix} \begin{bmatrix} \dot{\sigma}_1 \\ \dot{\sigma}_2 \\ \vdots \\ \dot{\sigma}_{n-1} \\ \dot{\sigma}_n \end{bmatrix} = \begin{bmatrix} \frac{\beta_1}{Ch} g - \frac{1}{2}\dot{f} \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix} \tag{45}$$

and

$$\frac{1}{h^2}\begin{bmatrix} 2 & -1 & & \\ -1 & 2 & -1 & \\ & - - - & - - - & \\ & & -1 & 2 \end{bmatrix} \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \vdots \\ \sigma_n \end{bmatrix} + \begin{bmatrix} 1 & & \\ & 1 & \\ & & - - - \\ & & & 1 \end{bmatrix} \begin{bmatrix} \ddot{\sigma}_1 \\ \ddot{\sigma}_2 \\ \vdots \\ \ddot{\sigma}_n \end{bmatrix} + \begin{bmatrix} 0 & 1 & & \\ & & 1 & \\ & & - - - & \\ & & & 1 \end{bmatrix} \begin{bmatrix} T_o \\ T_1 \\ - \\ T_N \end{bmatrix} = \begin{bmatrix} \frac{1}{h^2} f \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tag{46}$$

Equations (45) and (46) can be abbreviated as follows.

$$\frac{1}{Ch^2}[E_1]\underline{T} + (1+\lambda_1\lambda_2)[F_1]\dot{\underline{T}} + \lambda_1[G]\dot{\underline{\sigma}} = f_1$$

$$\frac{1}{h^2}[E_2]\underline{\sigma} + [F_2]\ddot{\underline{\sigma}} + \lambda_2[G]^T\ddot{\underline{T}} = f_2 \tag{47}$$

where $[E_1]$, $[E_2]$, etc. are the respective coefficient matrices in (45) and (46). Notice the equations in (45) are of the same format as those in (35), only the coefficient matrices are different.

THE TIME DOMAIN

Since Eqs. (35) and (47) are of the identical format, only (35) will be discussed in detail. Equation (35) can be further abbreviated as follows.

$$\begin{bmatrix} [0] & [0] \\ \lambda_2 [H]^T & [B] \end{bmatrix} \begin{bmatrix} \ddot{\underline{T}} \\ \ddot{\underline{\sigma}} \end{bmatrix} + \begin{bmatrix} (1+\lambda_1\lambda_2)\ [B_1] & \lambda_1 [H] \\ [0] & [0] \end{bmatrix} \begin{bmatrix} \dot{\underline{T}} \\ \dot{\underline{\sigma}} \end{bmatrix}$$

$$+ \begin{bmatrix} \frac{1}{Ch^2} [A_1] & [0] \\ [0] & \frac{1}{h^2} [A_1] \end{bmatrix} \begin{bmatrix} \underline{T} \\ \underline{\sigma} \end{bmatrix} = \begin{bmatrix} \underline{f_1} \\ \underline{f_2} \end{bmatrix} \qquad (48)$$

To further abbreviate, write (48) as

$$M\ddot{\underline{y}} + C\dot{\underline{y}} + K\underline{y} - \underline{F} = 0 \qquad (49)$$

where $\underline{y} = \begin{bmatrix} \underline{T} \\ \underline{\sigma} \end{bmatrix}_{(N+n)}$, other symbols are self-identifiable.

In discretizing the time domain we shall develop the recurrence schemes via i) Galerkin and ii) backward difference schemes.

Galerkin

Expanind $\underline{y}$ in an infinite series [6]

$$\bar{\underline{y}} = \sum_{i=0}^{\infty} N_i(t)\ \underline{y}_i$$

$$\text{where } N_i(t) \begin{cases} =1-\left|\dfrac{t-t_i}{\Delta t}\right|, & |t-t_i| \le \Delta t \\ =0, & |t-t_i| \ge \Delta t \end{cases}$$

Abbreviating (49) as $L(\underline{y}) = 0$ and define $R(\underline{y}) \equiv L(\bar{\underline{y}})$. Letting the weighted residue vanish and performing the necessary integrations, we have

$$\frac{M}{\Delta t}(\underline{y}_{i-1}-2\underline{y}_i+\underline{y}_{i+1}) + \frac{C}{2}(-\underline{y}_{i-1}+\underline{y}_{i+1}) + \Delta t\ \frac{K}{6}(\underline{y}_{i-1}+4\underline{y}_i+\underline{y}_{i+1})$$

$$= \frac{\Delta t}{6}(\underline{F}_{i-1}+4\underline{F}_i+\underline{F}_{i+1}) \qquad (51)$$

Rearranging,

$$(M+\frac{\Delta t}{2}C + \frac{\Delta t^2}{6})\underline{y}_{i+1} - (2M-\frac{2}{3}\Delta t K)\underline{y}_i + (M-\frac{\Delta t}{2}+\frac{\Delta t^2}{6}K)$$

$$= \frac{\Delta t}{6}(\underline{F}_{i-1}+4\underline{F}_i+\underline{F}_{i+1}) \qquad (52)$$

Approximating the First Step

For the recurrence scheme to proceed, both $\underline{y}_o$ and $\underline{y}_1$ must be known. An approximation of $\underline{y}_1$ is obtained by linear interpolation of the function $\underline{y}(t)$ over the interval, i.e.

$$\underline{y}(t) = N_o(t)\underline{y}_o + N_1(t)\underline{y}_1 \qquad (53)$$

Again requiring the weighted residue to vanish, i.e.

$$\int_0^{\Delta t} R\ N_1(t)dt = 0$$

we have, after a lengthy calculation,

$$(\frac{C}{2} + \frac{\Delta t}{3}K)\underline{y}_1 = (\frac{C}{2} - \frac{\Delta t}{6}K)\underline{y}_o + (\frac{\Delta t}{3})\ \underline{F}_1 \qquad (54)$$

Backward Difference

In this scheme a quadratic interpolation between two successive time increments is used and the weighting function is $\delta(t-t_{j+1})$.

The vanishing of the weighted residue yields

$$(M+\frac{3\Delta t}{2}C)\underline{y}_{i+1} - (2M+2\Delta t+C)\underline{y}_i + (M+\frac{\Delta t}{2}C+K)\underline{y}_{i-1} = \underline{F}_{i+1} \qquad (55)$$

COMPUTATION AND DISCUSSION

In the previous sections methodologies based on finite element approach have been developed to treat the boundary value problem of the coupled PDE of dynamic thermoelasticity. Some computational results will be discussed.

Different combinations of the spatial and time discretization were implemented. In the graphs presented at the end of this paper a notation such as "CD/BK" would mean central difference in spatial discretization and backward difference in time discretization. The notation GK stands for Galerkin.

The parameters of the mathematical model are only suggestive values which would provide some idea about a realistic problem. Within this objective, we have chosen the following set of parameters $\lambda_1 = 1.0$, $\lambda_2 = 10^{-3}$ (to 10^{-2}), $T_o = 100°F$, $\beta_1 = \beta_2 = .4$.

There are two scales of time for the thermoelastic problem. One time scale corresponds to the time of travel through the slab of a dilatational wave. The other is related to the time of diffusion phenomenon. The difference in magnitudes of these two times span five orders of magnitude. The wave travel time is unity in the non-dimensional time defined in Section 3. The "diffusion" time is about 10^5 non-dimensional units. Therefore, basically there exist two types of responses in this type of thermoelastic problem, which we shall call micro-time and macro-time behavior respectively. Thus, the time increments Δt used in the computation also differ by similar orders of magnitude.

Three test problems will be discussed: (1) responses to a unit-step stress $f(t)$, (2) responses to a unit temperature step $g(t)$, (3) responses to both a stress and a temperature pulse, simulating the gas pressure and temperature in a gun barrel.

A Modified Unit-Step Stress Input, $f(t)=1-e^{.1t}$

Uncoupled Problem. The theoretical response (solution to the wave equation) of a slab to a unit-step stress input imposed at $x = 0$ is given in Figure 3 by the rectangular waves for $x = 0.24$. This theoretical answer is used for comparing the accuracies of the responses to the modified step computed by the several combinations of Δt and Δx as exhibited in the tabulation in Figure 3. This figure shows that decreasing Δt and Δx improves the accuracy. The amplitude of the response using $\Delta x = .05$ and $\Delta t = .002$ matches the theoretical solution. Other combinations show deterioration of accuracy after one or two cycles.

Figure 4 shows the comparison of the quality of the computational results based on different discretization schemes. It is shown that CD/BK and GK/BK give quite close results. In other words, the spatial discretizations by CD and GK do not yield significantly different results. The backward scheme in time introduces some artificial "damping", causing inaccuracy, whereas the Galerkin scheme in time generates oscillations of the amplitude about the exact value. These oscillations can be reduced by decreasing both Δt and Δx. Figure 4 shows that using $\Delta x = .05$ and $\Delta t = .02$ reduces the oscillations when compared to larger Δx and Δt. The graphs showing the results of larger increments are omitted for the sake of space.

Coupled Problem. Figure 5 shows the stress and the temperature response due to a modified unit stress applied at $x = 0$ and zero temperature input at $x = 0$ with coupling parameters $\lambda_1 = .01$ and $\lambda_2 = .1$. It is shown the propagation of the stress wave and also the temperature wave due to the coupling effect. The solid lines show the wave front at various time instants. The dotted lines show the temperature wave being synchronous with the stress wave. The magnitude of the non-dimensional temperature is 1/10 of that of the stress. This ratio is clearly determined by the coupling parameter λ_1 (=0.1). It is also observed that at the time corresponding to that of the wave travel through the slab (micro-time), no diffusion effect is observed.

A Modified Step Temperature Input, $g(t) = 30(1-e^{.1t})$

Uncoupled Problem. Figure 6 shows the temperature profile for various times due to an input of a modified temperature step with amplitude 30, for the uncoupled equation ($\lambda_1 = \lambda_2 = 0$). Since the thermal problem has a long time scale, the result demonstrates that using $\Delta t = 20$ and 500 yields no significant difference. Figure 7 shows the time response at two different field points. On the same figure the instability of the Galerkin time discretization is demonstrated.

Coupled Problem. Figure 8 and 9 are the results in responses for the coupling specified by $\lambda_2 = .01$ and two values of $\lambda_1 = 0.1$ and 10.0. It is observed that the temperature responses show significant difference when the coupling parameter λ_1 is 10.0, which is an upper bound value.

Dual Pulses, $f(t) = .1\ te^{-.1t}$, $g(t) = 8.1548\ te^{-.1t}$

Simultaneous inputs of gas pressure and temperature as approximated by the given set exponential functions yield responses at $x = .2$ in micro-time as shown in Figure 10. It is observed that the wave front arrives at $x = .2$ at $t = .2$. The successive cycles of wave travel are depicted by the ripples oscillating about a mean curve following the general input pulse wave form.

Figure 11 shows the micro-time response in the temperature and the stress. For $\lambda_1 = 1.0$ and $\lambda_2 = .01$ the top figure shows that the coupling effect on stress is minimal, whereas the lower figure shows that the coupling has a strong effect on the wave-like temperature profile.

Figure 12 and 13 show the macro-time temperature vs. time and the temperature profile under the action of a dual stress and temperature input. Notice that in macro-time the stress remaining is low. This is due to the fact that the slab is free to expand.

CONCLUSION

The theory and the implementation of a finite element methodology in solving the problem of the couple dynamic thermoelastic slab has been established. Test problems of a unit-step stress input as well as a unit-step temperature input are used to exhibit the two different types of responses, namely, the micro- and the macro-time behavior of the slab. In the spatial discretization both the Galerkin scheme and the central difference are satisfactory whereas the backward difference scheme is preferred in the time discretization. A realistic set of the stress and the temperature inputs, simulating the interior ballistics of a gun barrel, is used to generate responses.

ACKNOWLEDGMENT

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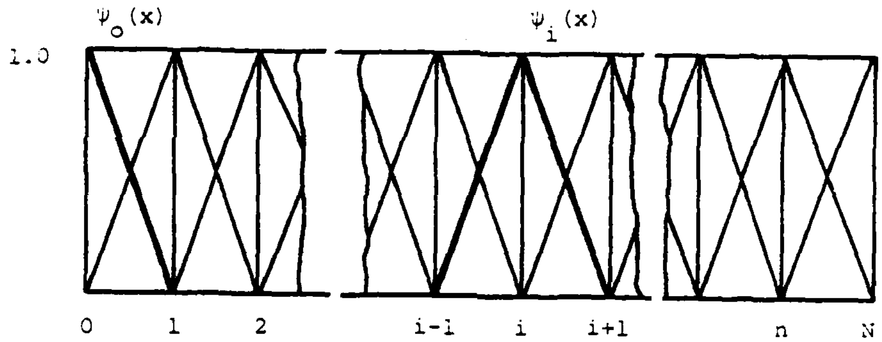

Figure 1. - Galerkin

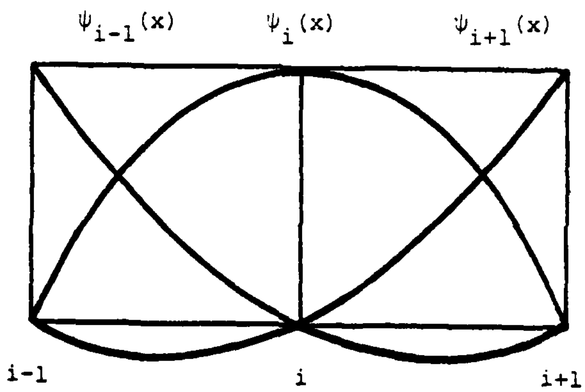

Figure 2. - Central Difference

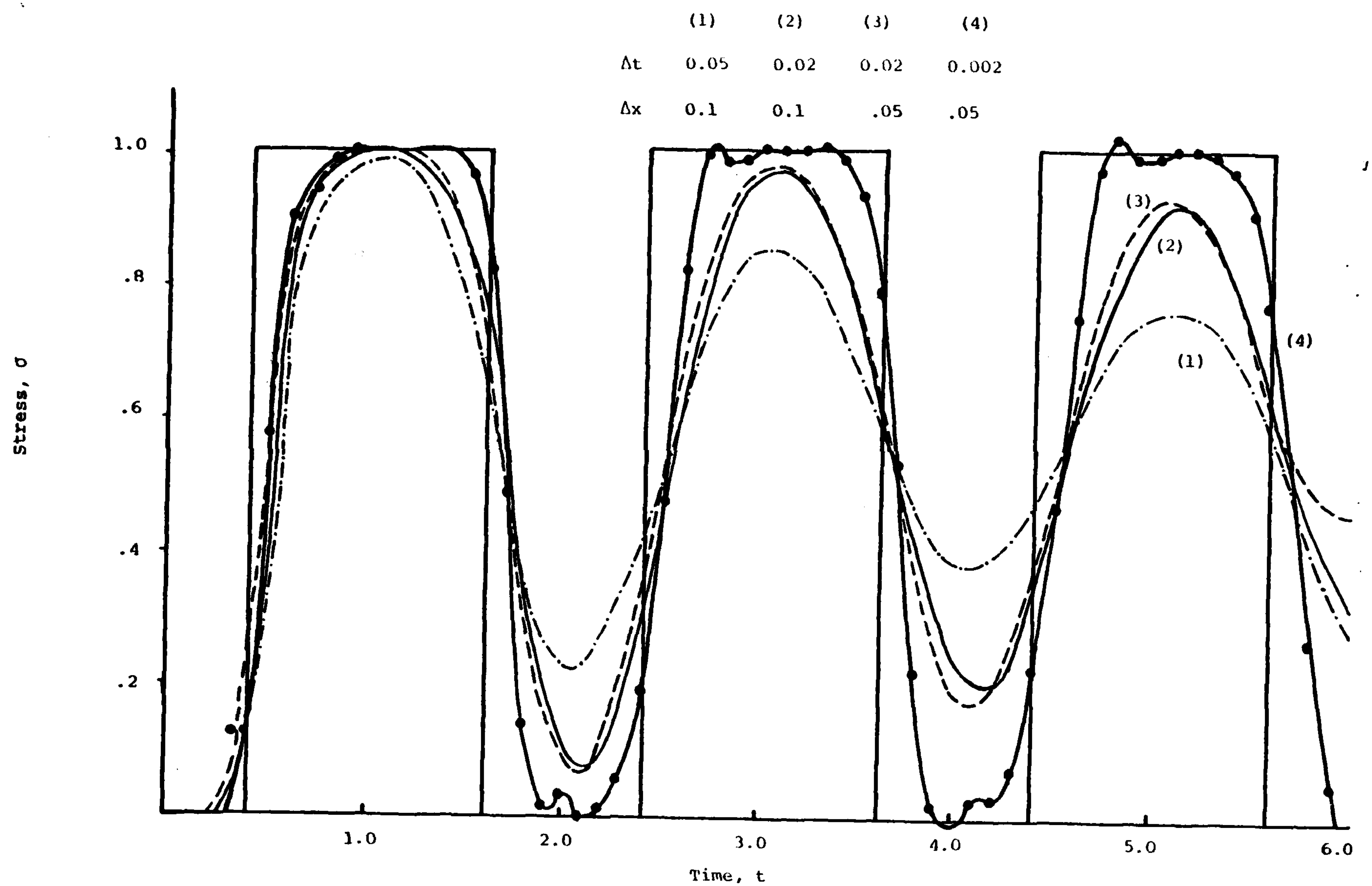

Figure 3. - Stress vs. Time at x = .4, Step-Stress at x = 0 (CD/BK)

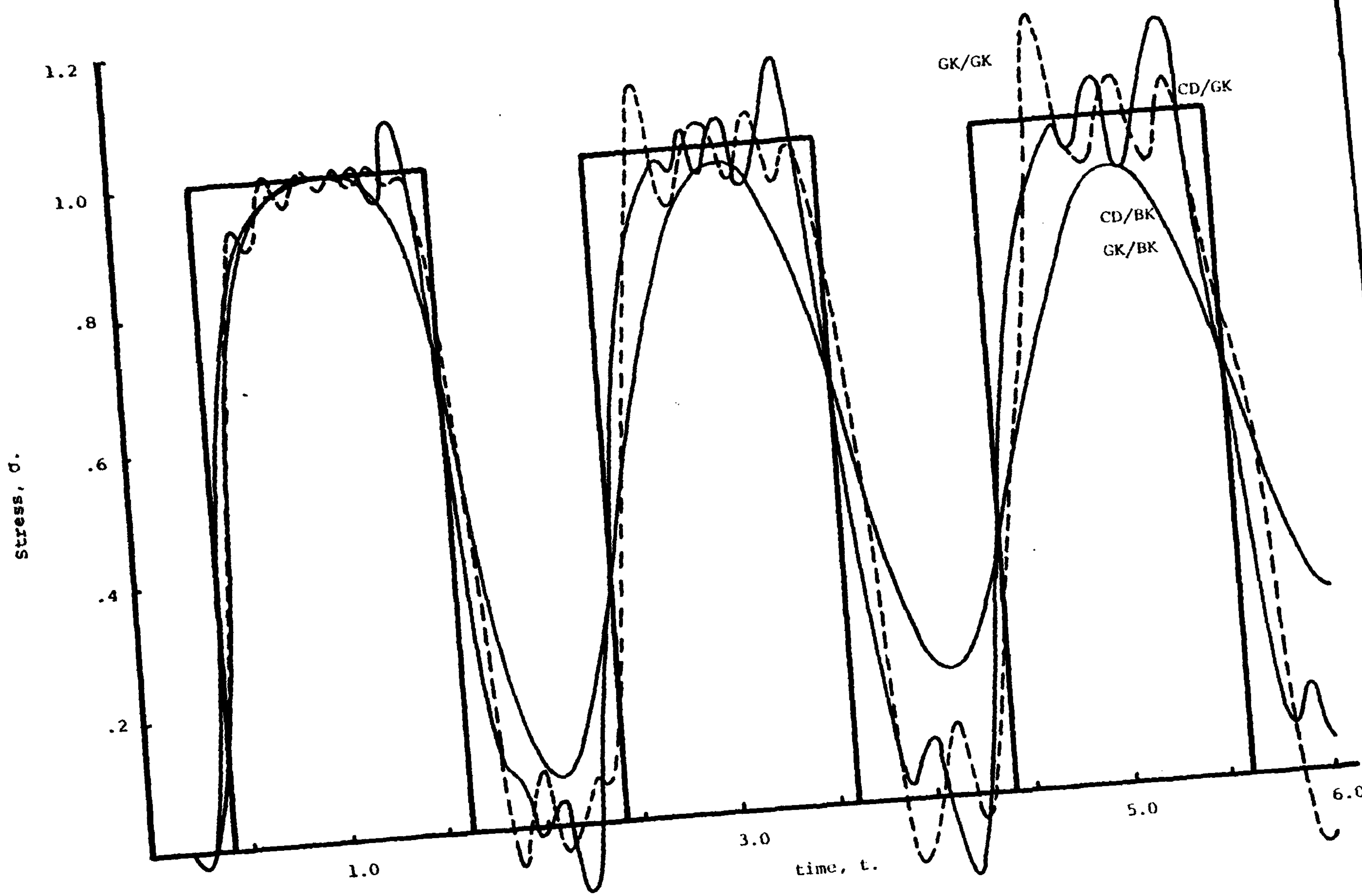

Figure 4. Stress vs. Time by Different Schemes

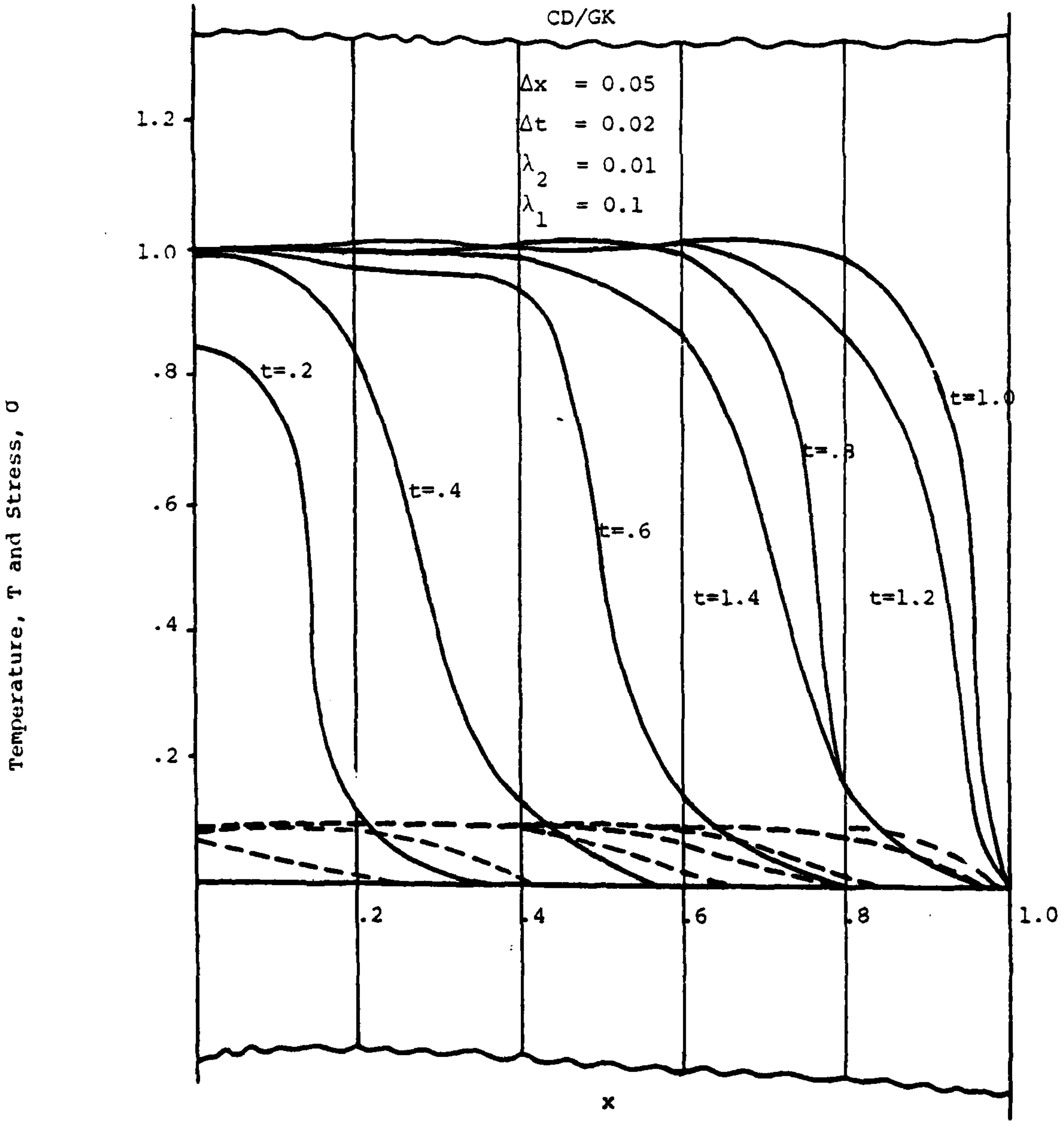

Figure 5. - Stress and Temperature vs x
Unit stress at x = 0 for various t

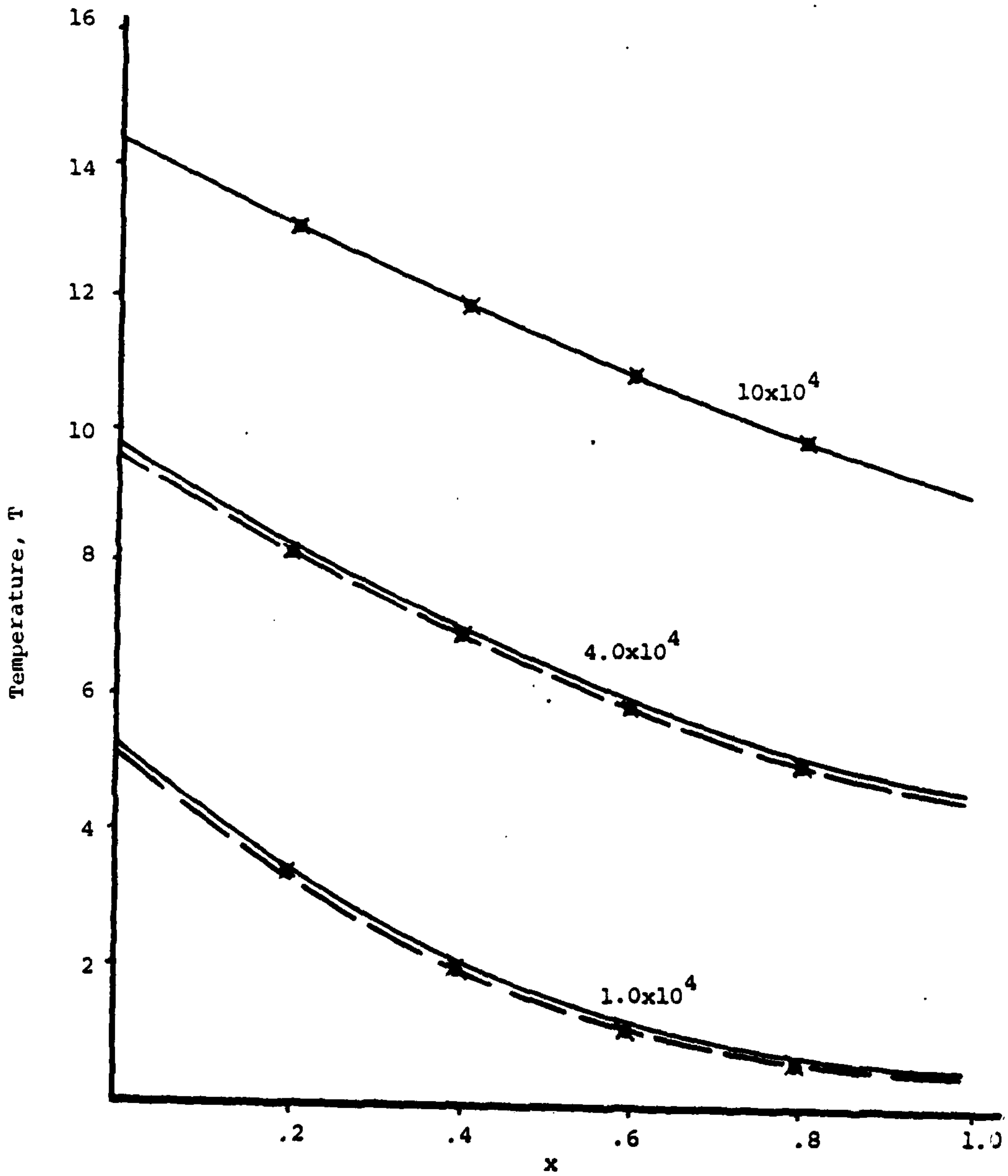

Figure 6. - Temperature Profiles ($\lambda_1 = \lambda_2 = 0$) for Input $g = 30(1-e^{-.1t})$

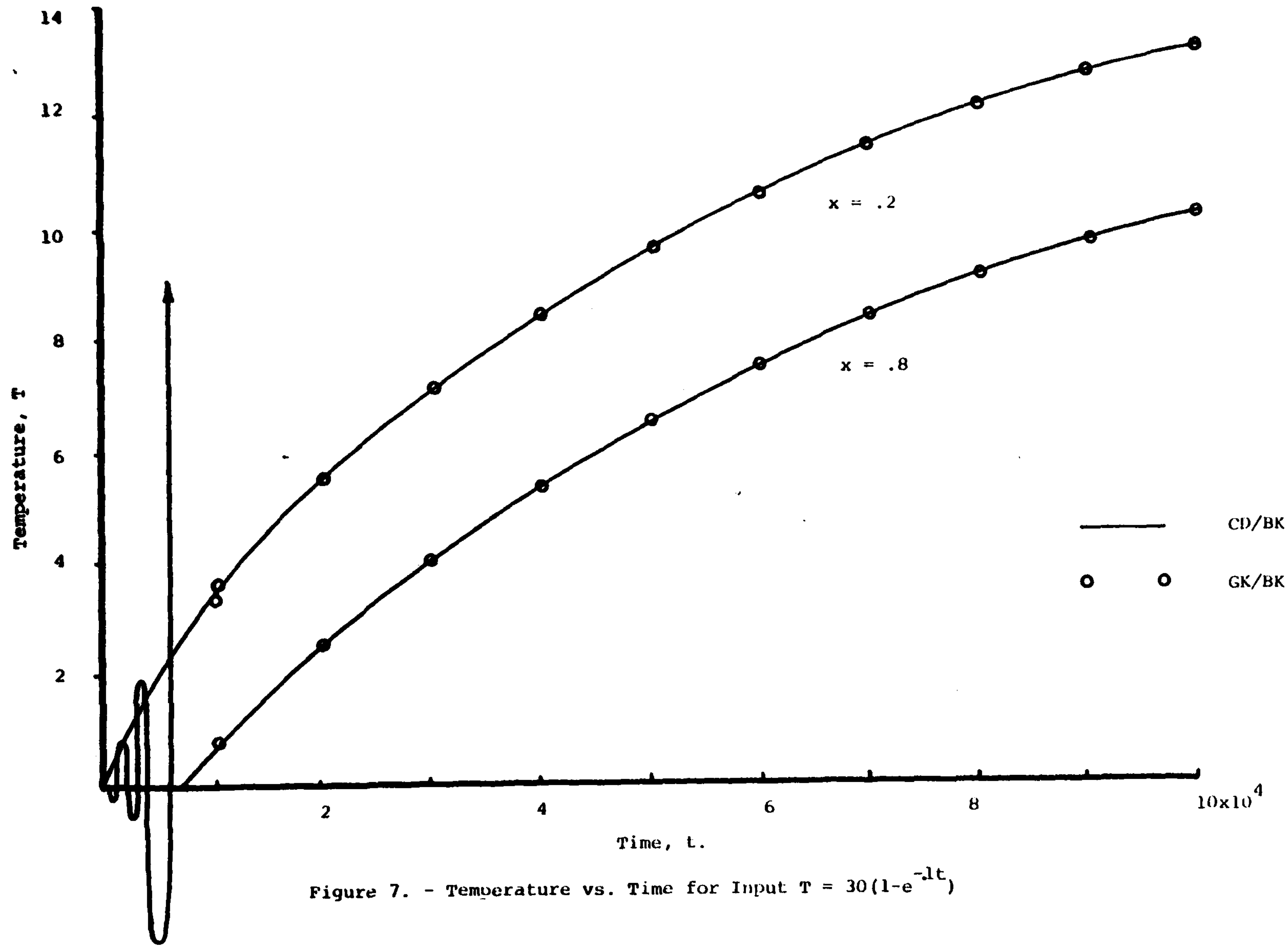

Figure 7. - Temperature vs. Time for Input $T = 30(1-e^{-.1t})$

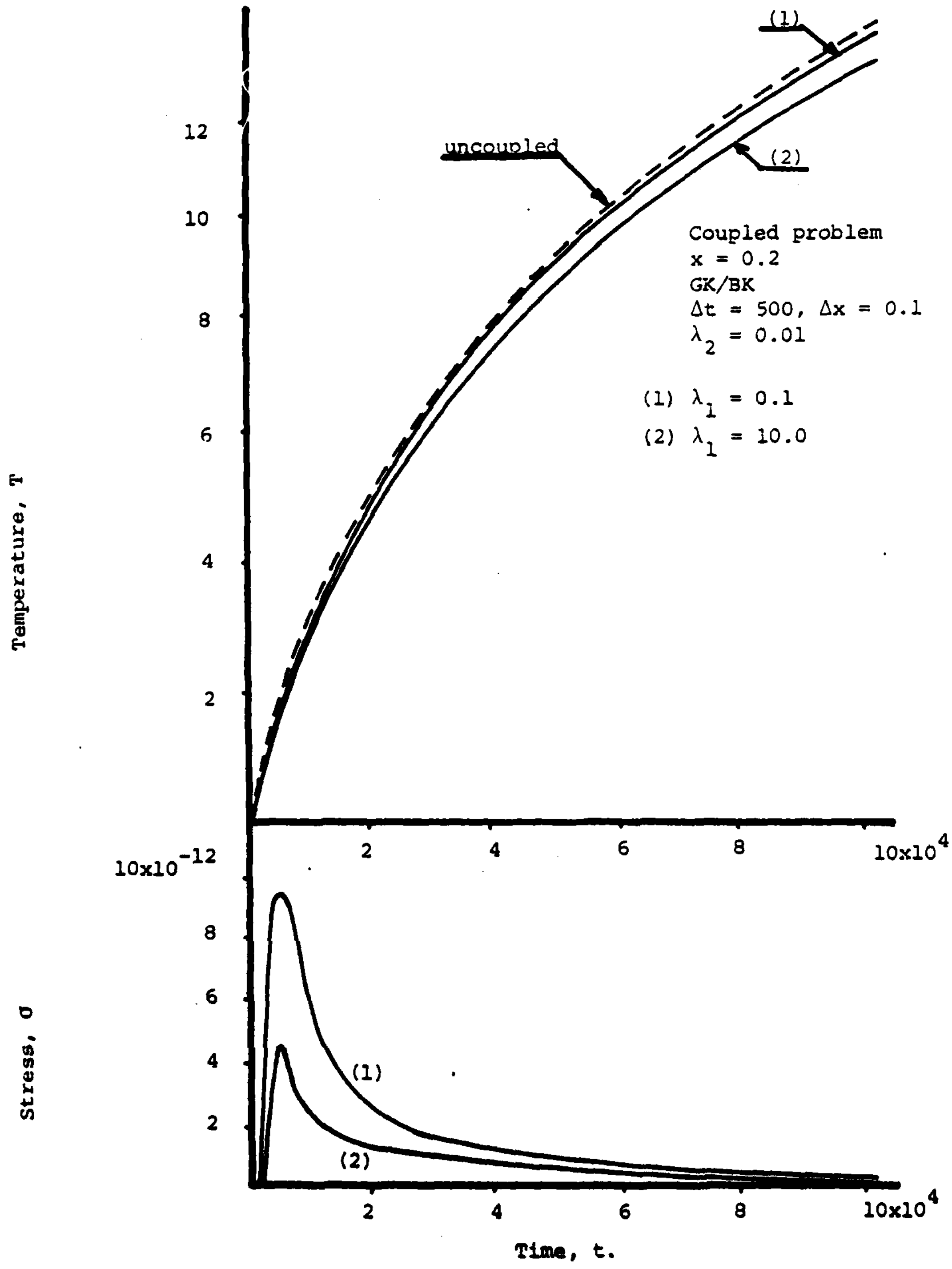

Figure 8. - Temperature and Stress vs. Time

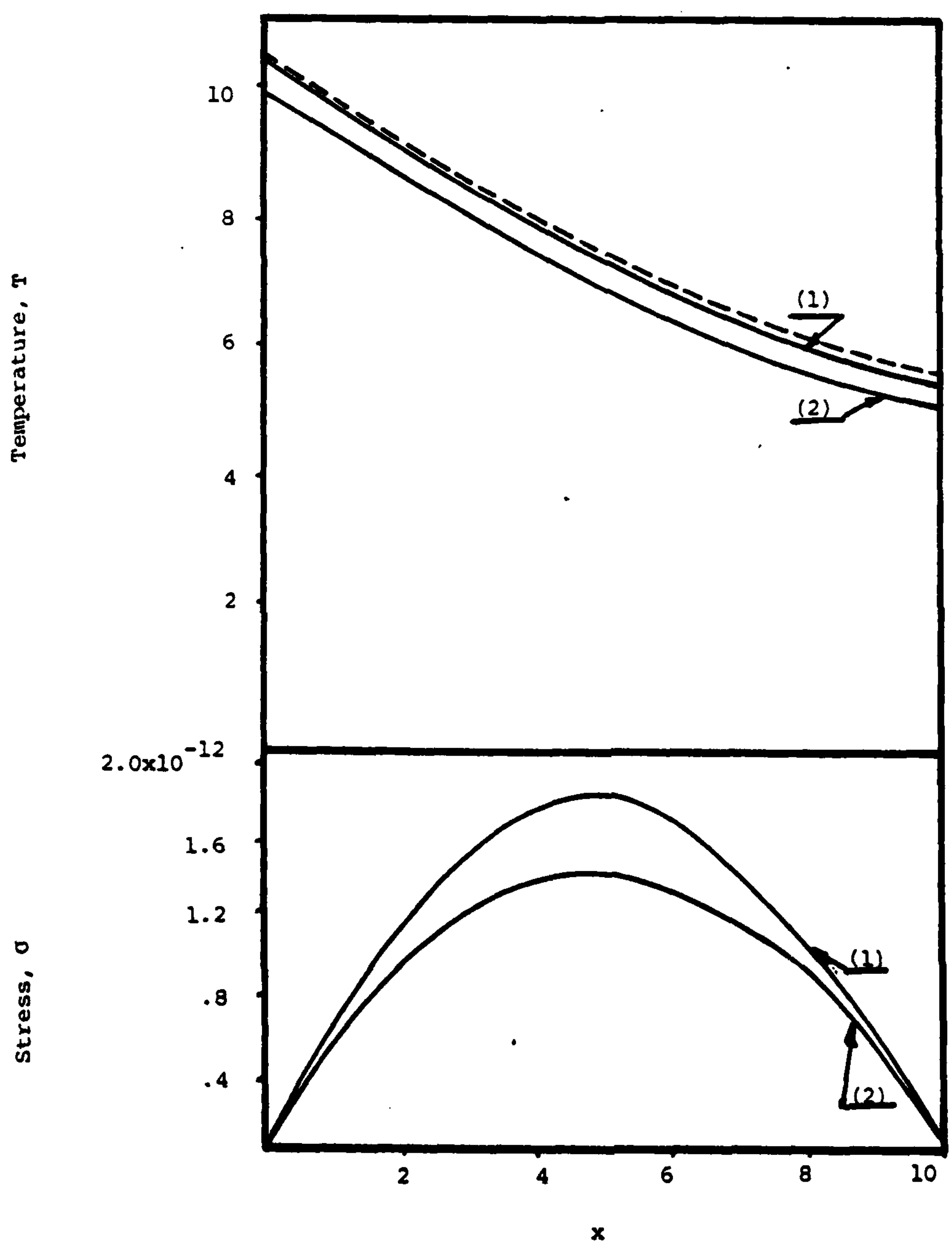

Figure 9. - Temperature Profile

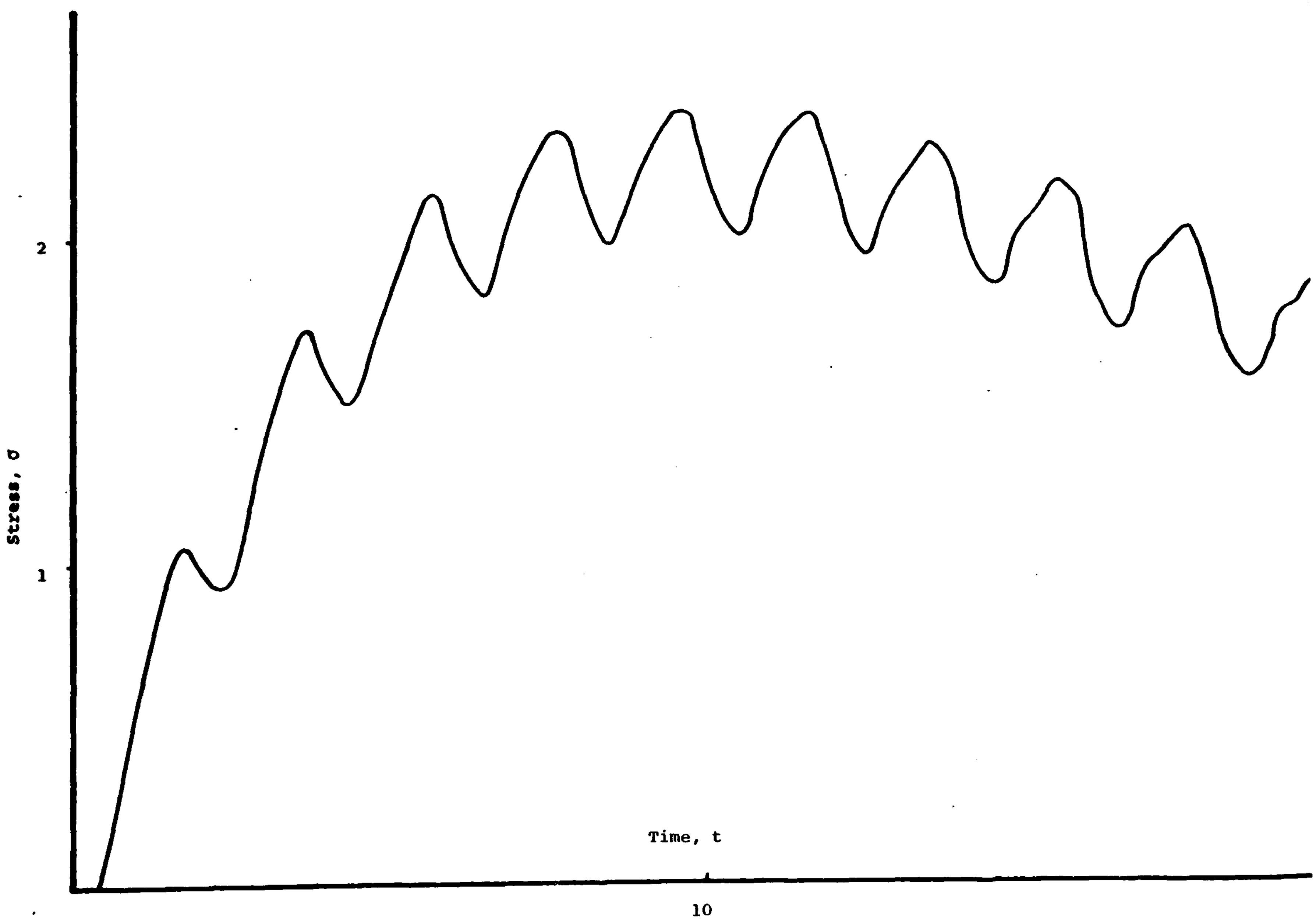

Figure 10. - Micro-time Stress Response

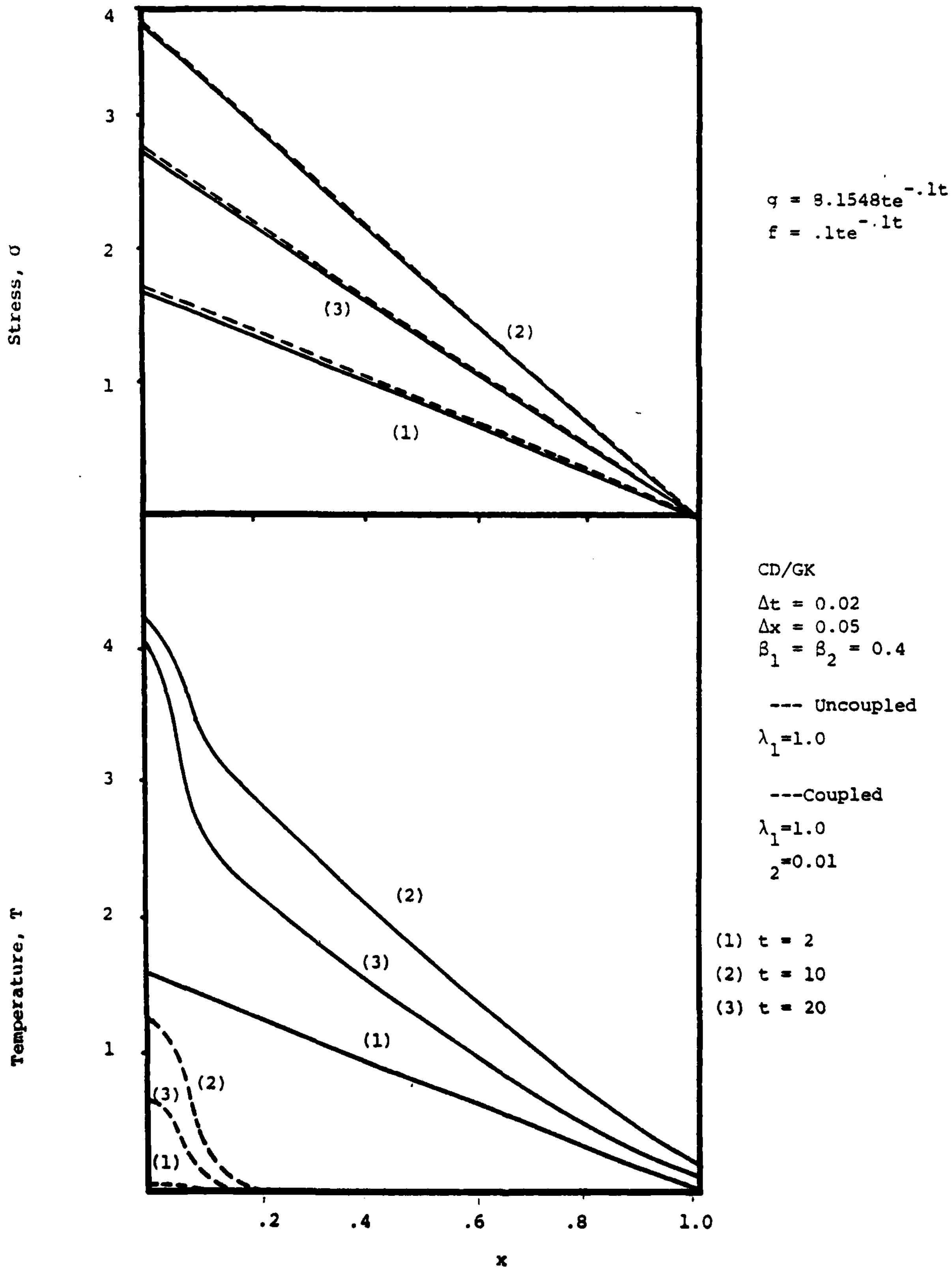

Figure 11. - Temperature and Stress Profile

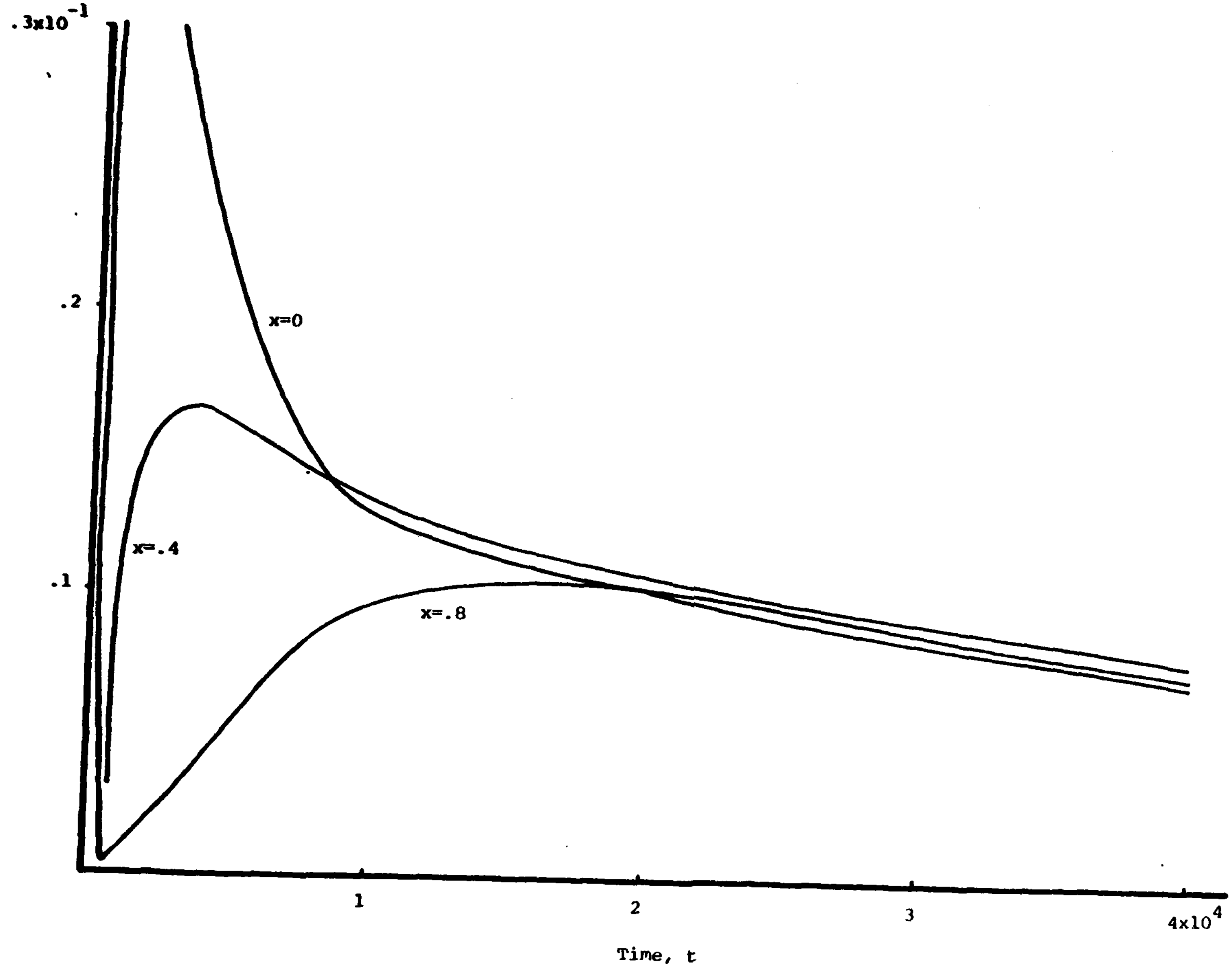

Figure 12. - Temperature vs. Time

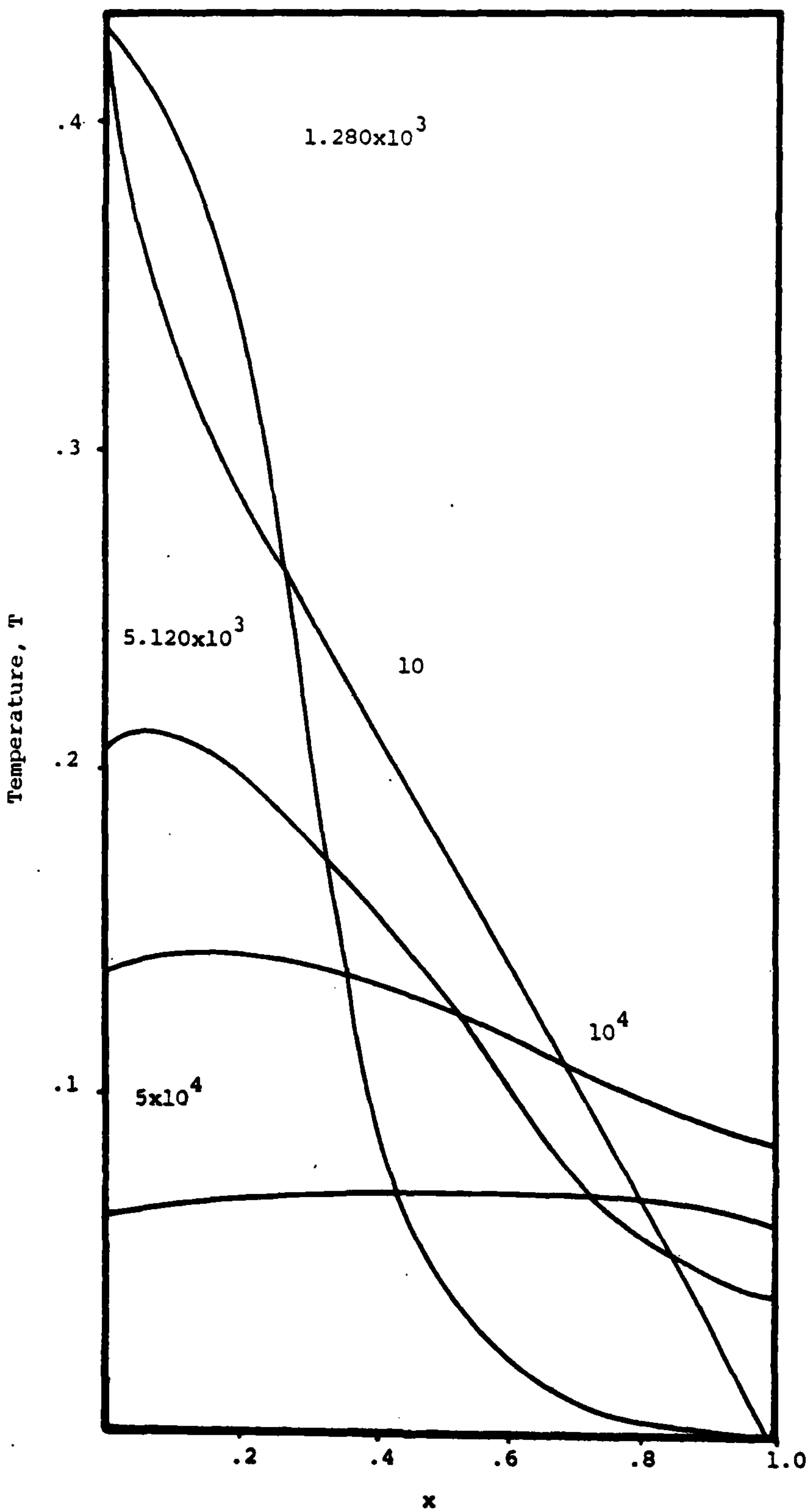

Figure 13. - Temperature Profile